

A UNIFIED THEORY OF CREEP BUCKLING OF COLUMNS, PLATES, AND SHELLS*

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ABSTRACT

Creep-buckling theory of perfect columns is first developed using an equation of state for the incremental stresses that arise at buckling. The theory is then developed in a general two-dimensional form for analysis of plates and shells. It is proved that creep-buckling solutions are directly analogous to inelastic-buckling solutions provided that the tangent and secant moduli are treated as strain-rate dependent quantities. Thus, the creep-buckling results apply directly to any arbitrary compressive creep data without requiring a specific creep law. The theory is correlated with available test data on columns, flat plates, and shells.

INTRODUCTION

Creep-buckling theories have evolved along two main lines that are representative extensions of approaches to short-time elastic and inelastic-stability analyses. The initial imperfection approach to creep buckling postulates that initial imperfections in geometry or loading grow with time ultimately leading to failure. The classical stability approach hypothesizes that an exchange of stable equilibrium configurations from the straight to the bent form occurs at a critical time at which compressive creep has resulted in reductions in flexural and extensional rigidities.

For short-time stability, the simplicity and predictive value of classical stability theory has favored its use over the more detailed calculations required by the initial imperfection theories. For the latter, not only must the value of initial imperfection be known or assumed, but the development is further

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encumbered by the necessity of assuming an explicit form of variable stress creep law. The classical stability approach, on the other hand, has the great advantage of not requiring specification of the degree of initial imperfection and further does not require the assumption of an explicit creep law.

A satisfactory classical stability theory for creep buckling of perfect columns and plates was first developed by Rabotnov and Shesterikov.¹ Their analysis of creep buckling is of major importance because it is based upon fundamental concepts that logically extend classical stability concepts into the creep regime. Among the concepts utilized are a mechanical equation of state to represent the time-dependent behavior at instability and a time-dependent formulation of the governing differential equations and stability criterion.

In the present paper, certain modifications of the original concepts are introduced and the theory is extended to creep buckling of perfect shells. Of particular importance is a proof that the derived creep-buckling solutions for arbitrary creep properties are directly equivalent to inelastic buckling solutions provided the tangent and secant moduli are strain-rate dependent quantities.

SYMBOLS

- a = Plate length
- A_i = Plasticity coefficients
- b = Plate width
- B = Extensional rigidity, $B = E_s h / (1 - \nu^2)$
- D = Flexural rigidity, $D = E_s h^3 / 12(1 - \nu^2)$
- E = Elastic modulus
- $[E_r]_i$ = Strain-rate dependent reduced modulus
- E_s = Secant modulus
- $[E_s]_i$ = Strain-rate dependent secant modulus
- E_t = Tangent modulus
- $[E_t]_i$ = Strain-rate dependent tangent modulus
- $[E_u]_i$ = $\partial \sigma_i / \partial \dot{\epsilon}_i$
- h = Thickness
- k = Buckling coefficient
- L = Column length
- t = Time
- T = Temperature
- w = Lateral deflection
- x, y, z = Coordinates
- γ = Shear strain
- ϵ = Normal strain
- ϵ_c = Creep component of strain
- $\dot{\epsilon}$ = Strain rate
- η = Creep buckling reduction factor
- ν_e = Elastic Poisson's ratio
- ν = Plastic Poisson's ratio
- ρ = Radius of gyration

- σ = Normal stress
- σ_a = Applied creep stress
- τ = Shear stress

Subscripts

- i = Intensity
- x, y, z = Coordinate orientation
- $\dot{}$ = Time derivative

BASIC CONCEPTS

It is hypothesized that a perfect column, plate or shell acting under in-plane loads undergoes corresponding deformations before buckling occurs, and therefore follows the appropriate constant in-plane stress creep relation. At buckling, an exchange of equilibrium configurations occurs from the straight to the laterally deflected form. Hence, a variable stress creep law is required to relate the incremental stresses and strains associated with lateral bending in the presence of a relatively large in-plane strain.

The essential difference between creep buckling and inelastic buckling results from the fact that the *constant* stress state existing prior to creep buckling produces time-dependent deformations whereas the *increasing* stress state associated with inelastic buckling produces time-independent deformations. For creep buckling, it is only the reduction of the extensional and flexural rigidities with time under a constant stress that can account for buckling of perfect structural elements.

CREEP PROPERTIES

Rabotnov and Shesterikov¹ have assumed that a mechanical equation of state in terms of the creep components of strain, $(\epsilon_c)_i$, may be used in the following form,

$$\varphi [\sigma_i, (\epsilon_c)_i, (\dot{\epsilon}_c)_i, T] = 0 \tag{1}$$

Equation (1) would seem to be valid for small departures from the initial in-plane stress state, the conditions which exist in the creep-buckling problem. It appears, however, that neglect of the elastic strain and strain rate components may not be justified particularly in the region of primary creep. In fact, many of the phenomenological creep relations currently in favor are in terms of the total strain rate, $\dot{\epsilon}_i$. As a consequence, the following equation of state has been assumed in the analysis presented herein.

$$\varphi (\sigma_i, \epsilon_i, \dot{\epsilon}_i, T) = 0 \tag{2}$$

In general form, the stress intensity, σ_i , the strain intensity, ϵ_i and strain rate intensity, $\dot{\epsilon}_i$, are defined according to the familiar plasticity relations for plane stress:

$$\sigma_i = (\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau^2)^{1/2} \tag{3}$$

$$\epsilon_i = [2/(3)^{1/2}](\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \gamma^2/4)^{1/2} \quad (4)$$

$$\dot{\epsilon}_i = [2/(3)^{1/2}](\dot{\epsilon}_x^2 + \dot{\epsilon}_y^2 + \dot{\epsilon}_x\dot{\epsilon}_y + \dot{\gamma}^2/4)^{1/2} \quad (5)$$

Equation (2) implies that creep data may be transformed into constant strain rate stress-strain data in terms of σ_i , ϵ_i and $\dot{\epsilon}_i$ in the form shown schematically in Fig. 1 for a constant temperature.

In buckling of plates and shells, the secant modulus is associated with the extensional rigidity. In Fig. 1, the secant modulus for a prescribed value of strain rate intensity is defined as

$$[E_s]_{\dot{\epsilon}_i} = \sigma_i / \epsilon_i \quad (6)$$

Thus, for given values of σ_i and $\dot{\epsilon}_i$, $[E_s]_{\dot{\epsilon}_i}$ uniquely defines a point on the φ -surface of Fig. 1.

For an incremental departure from the point on this surface representing the bent state in the buckling process

$$\delta\sigma_i = (\partial\sigma_i/\partial\epsilon_i)\delta\epsilon_i + (\partial\sigma_i/\partial\dot{\epsilon}_i)\delta\dot{\epsilon}_i \quad (7)$$

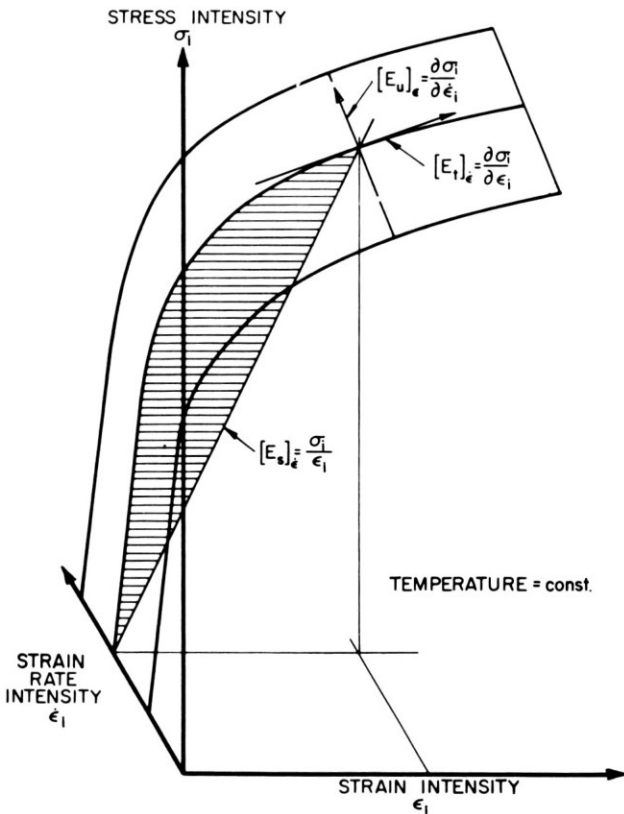


Fig. 1. Stress-strain-strain rate conditions associated with creep buckling.

The tangent modulus associated with the flexural rigidity is defined as

$$E_t = \delta\sigma_i / \delta\epsilon_i \tag{8}$$

Consequently, by use of Eq. (8), Eq. (7) becomes

$$E_t = \partial\sigma_i / \partial\epsilon_i + (\partial\sigma_i / \partial\dot{\epsilon}_i)(\delta\dot{\epsilon}_i / \delta\epsilon_i) \tag{9}$$

In a more convenient form

$$E_t = [E_t]_{\dot{\epsilon}} + [E_u]_{\epsilon} (\partial/\partial t) \tag{10}$$

BUCKLING MODELS

While Eqs. (2) to (10) are important in the creep buckling of columns, plates and shells, we shall first consider the column in some detail because of its fundamental significance. Before proceeding to the formal analysis, however, it seems worthwhile to review the buckling models that have been accepted for elastic and inelastic stability of columns. From these basic principles, the creep buckling mechanism can be shown to evolve logically.

The fundamental perfect column problem is the determination of the effective bending stiffness after the column has undergone end shortening whether it be elastic, inelastic or creep. Consequently, it is necessary to examine the stress-strain path associated with axial deformation as distinct from that associated with incremental bending. For the simplest case, elastic buckling, the axial deformation follows path *OA* as shown in Fig. 2a; incremental bending follows *AA+* on the concave side and *AA-* on the convex side. Thus, the elastic modulus is associated with the bending stiffness.

The commonly accepted model for inelastic buckling is illustrated in Fig. 2b. According to the tangent modulus model, axial deformation proceeds along path *OA* until buckling occurs. At buckling, incremental bending and axial loading proceed simultaneously so that the stress state on the convex side remains stationary at *A* while the concave side follows path *AA+* in the direction of the local tangent modulus.

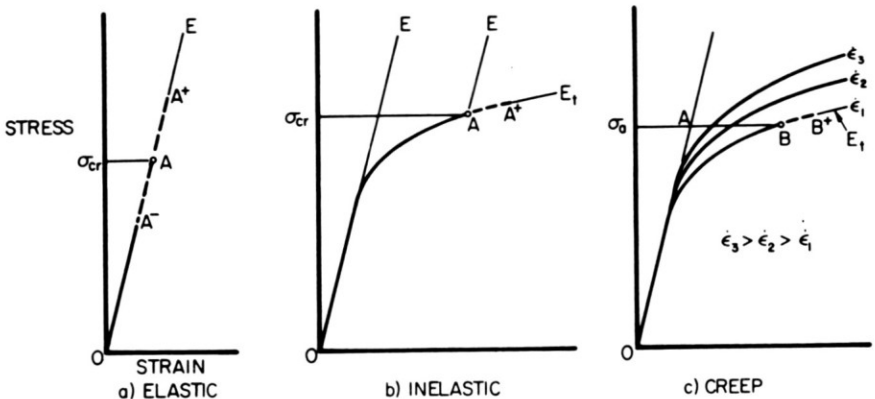


Fig. 2. Models of buckling behavior.

The last statement implies an assumption concerning the strain rate of infinitesimal bending that was first pointed out in Ref. 2. In order for the path $AA+$ to be in the direction of the local tangent modulus, loading on the concave side must proceed at a strain rate associated with the compressive stress-strain curve. If this assumption were not implied, but instead instantaneous incremental bending were assumed, then the local bending stiffness would have a value of E and buckling would not occur. Thus, incremental bending at the local strain rate leads to a lower limit to the bending stiffness which appears to be the appropriate value to use in determining instability. Since the conditions at inelastic buckling deviate only slightly from those before buckling, the value of the tangent modulus is governed by the axial compressive conditions of σ_i , ϵ_i and $\dot{\epsilon}_i$.

In creep buckling, a series of stress-strain curves at different strain rates reflect the time-dependent nature of the problem as shown in Fig. 2c. In the presence of an applied compressive stress, it was pointed out in Ref. 2 that the local tangent modulus decreases with time as successive strain-rate curves are crossed along path AB until buckling occurs. Thus, for creep buckling as for inelastic buckling the lower bound to the bending stiffness is taken as the tangent modulus appropriate to the strain-rate conditions at buckling. In this sense, the creep-buckling mechanism forms a logical extension of that associated with inelastic buckling.

Shanley³ has pointed out that even in the presence of a constant axial load, σ_a , the tangent modulus model can be used by assuming that the axial strain increases slightly at buckling to prevent a strain reversal on the convex side of the column. Thus, no part of the column unloads and it is not necessary to use the reduced modulus concept or account for creep recovery effects at buckling.

CREEP BUCKLING OF COLUMNS

Rabotnov and Shesterikov¹ considered the stability of a column subject to creep buckling from both a dynamic and quasi-static point of view with the same result. In the dynamic analysis, the pertinent equations of motion were derived and the stability of the system was determined from the damping characteristics of an infinitesimal lateral oscillation. In the quasi-static treatment, a time-dependent equilibrium equation was formulated and the stability was determined from the character of the lateral deflection following the removal of an infinitesimal lateral disturbance. The essential feature of the creep-buckling problem which appears in its simplest form in the quasi-static analysis of Ref. 1 is used in the following together with the equation of state in terms of the total strain, Eq. (2) rather than Eq. (1) which was used in Ref. 1.

THEORETICAL DEVELOPMENT

With the assumption that plane sections remain plane, the incremental bending strain that arises at buckling is given by

$$\delta\epsilon_i = -z\partial^2 w/\partial x^2 \quad (11)$$

By use of Eqs. (7) and (10), Eq. (11) becomes

$$\delta\sigma_i = - [E_t]_i z (\partial^2 w / \partial x^2) - [E_u]_{i\epsilon} z (\partial^3 w / \partial x^2 \partial t) \tag{12}$$

Since $[E_t]_i$ and $[E_u]_{i\epsilon}$ remain constant in the presence of an axial compressive stress, Eq. (12) can be integrated directly. From equilibrium between the external bending moment and the internal bending resistance, the following governing equilibrium differential equation is obtained:

$$\frac{\partial^2 w}{\partial x^2} + \frac{[E_u]_{i\epsilon}}{[E_t]_{i\epsilon}} \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\sigma_a}{[E_t]_{i\epsilon} \rho^2} w = 0 \tag{13}$$

A solution of Eq. (13) can be given in the following form for a simply supported column:

$$w = w_0 [t] \sin (\pi x / L) \tag{14}$$

By substituting the appropriate derivatives of Eq. (14) into (13),

$$\left[- \left(\frac{\pi}{L} \right)^2 w_0 - \frac{[E_u]_{i\epsilon}}{[E_t]_{i\epsilon}} \left(\frac{\pi}{L} \right)^2 \dot{w}_0 + \frac{\sigma_a}{[E_t]_{i\epsilon} \rho^2} w_0 \right] \sin (\pi x / L) = 0 \tag{15}$$

For a nontrivial solution, the bracketed terms are set equal to zero with the following result:

$$\frac{\sigma_a}{[E_t]_{i\epsilon} (\pi \rho / L)^2} = 1 + \frac{[E_u]_{i\epsilon}}{[E_t]_{i\epsilon}} \frac{\dot{w}_0}{w_0} \tag{16}$$

In order to complete the problem it is now necessary to introduce a stability criterion associated with the lateral deflectional response w_0 / w_0 .

STABILITY CRITERION

The stability investigation of an elastic or inelastic column consists of applying an infinitesimal lateral disturbance at the critical load to determine if the bent form vanishes or remains upon removal of the disturbance. For the creep-buckling case, the lateral disturbance is time dependent according to Eq. (14) and therefore Rabotnov and Shesterikov introduced a revised stability criterion.

As indicated schematically in Fig. 3, three paths are possible immediately after the infinitesimal lateral disturbance is removed. Of particular importance is the local tangent at w_0 represented by w_0 . For the stability problem, the neutral case $w_0 = 0$ governs and therefore in Ref. 1, the condition $w_0 / w_0 = 0$ was taken as the stability criterion for creep buckling. Thus, by use of this stability criterion Eq. (16) reduces to the following quasi-static form:

$$\sigma_a = [E_t]_i (\pi \rho / L)^2 \tag{17}$$

It is interesting to note that recently Shesterikov⁴ apparently rejected this stability criterion because of “. . . the contradiction which occurs in the quasi-static study, that with time the initial deformation of the compressed bar may

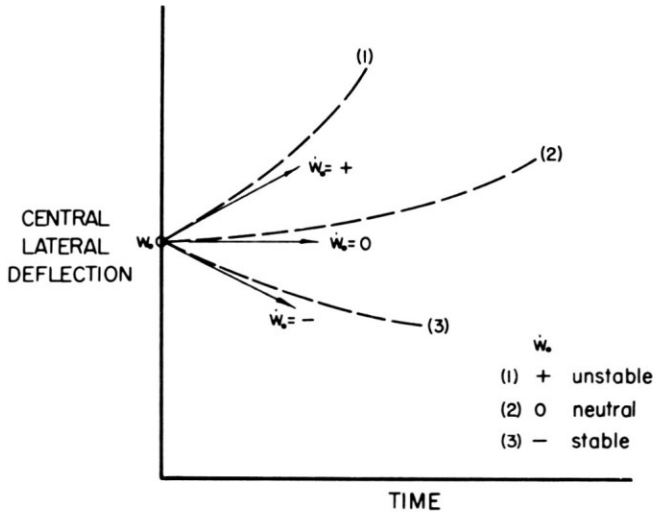


Fig. 3. Nature of the stability criterion.

decrease." This implies that in Fig. 3 only paths 1 and 2 are possible. In the treatment used herein which is in terms of the total strain rather than creep strain, it is apparent that upon the removal of the infinitesimal lateral disturbance recovery of the elastic strain component can result in path 3. Since only the local tangent at w_0 is of interest, all three paths are possible and the apparent contradiction is removed when the total strains are considered.

RESULTS OF THEORY

From Eq. (17), it is evident that since σ_a and L/ρ are prescribed, creep buckling of the column occurs when $[E_t]_e$ has reduced sufficiently through creep to satisfy this relationship. The critical time for creep buckling can be obtained from the following numerical procedure. For illustrative purposes, compressive creep data on aluminum alloy 2024-0 at 500°F obtained in the experimental program of Ref. 5 are used.

Tangents to the compressive creep curves of Fig. 4 represent the instantaneous values of strain rate. The strain rates for each creep curve may now be plotted as a function of strain as shown in Fig. 5. The intersections of the curves with any vertical in Fig. 5 represent the stress and strain data at a constant strain rate. It is now possible to construct constant strain rate stress-strain curves for any specified strain rate using these data as shown in Fig. 6.

In order to apply Eq. (17) conveniently, the tangent moduli to the curves given in Fig. 6 are determined, and plotted against stress as shown in Fig. 7. A straight line through the origin is associated with a particular L/ρ ratio for the column. Each intersection of such a line with a constant strain rate tangent modulus curve represents a unique set of conditions for creep buckling.

In order to predict creep-buckling times, one first finds the appropriate strain rate for the applied stress and L/ρ value from Fig. 7. The creep strain corresponding to the strain rate is then found from Fig. 5 and the creep-buckling time can be read off from the proper compressive creep curve in Fig. 4. Using this process, critical times were found for a number of applied stresses and L/ρ values and these results are given finally in Fig. 8 as the curve marked $[E_t]_c$.

It is important to note that the creep-buckling results presented in Fig. 8 represent the critical times at which lateral deflections first develop in a perfect column. Collapse of the column is of course a later event in the same sense short-time buckling and failure are distinctly different phenomena.

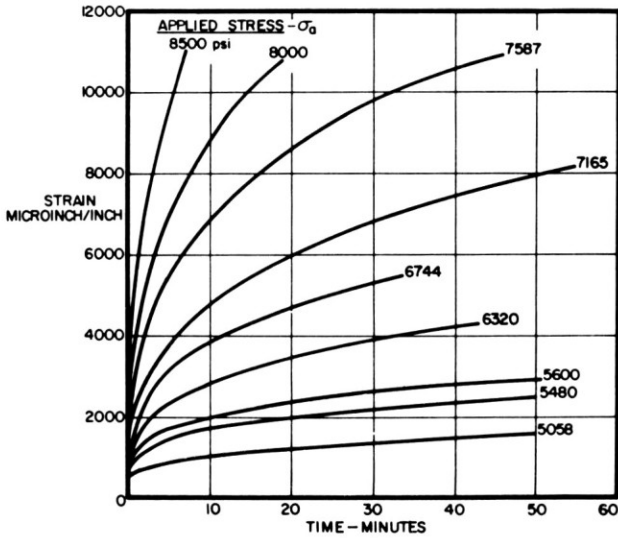


Fig. 4. Average compressive creep data for 2024-0 aluminum alloy at 500° F.

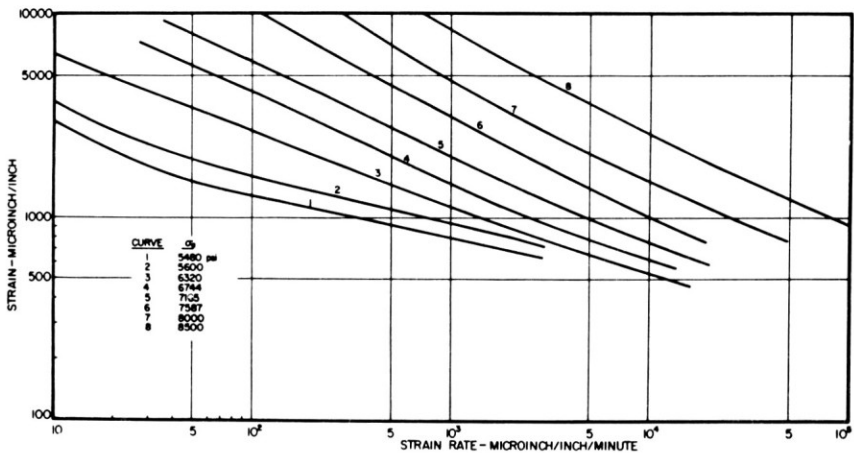


Fig. 5. Strain-strain rate data for 2024-0 aluminum alloy at 500° F.

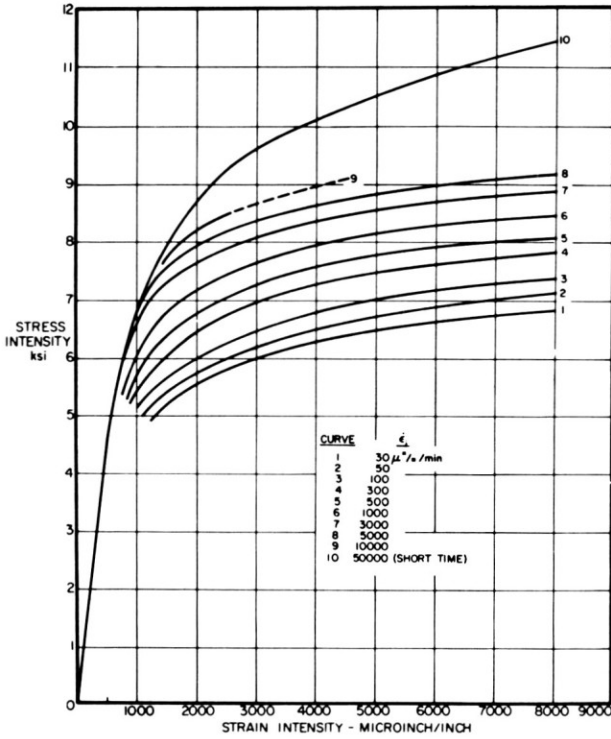


Fig. 6. Constant strain rate stress-strain data for 2024-0 aluminum alloy at 500° F.

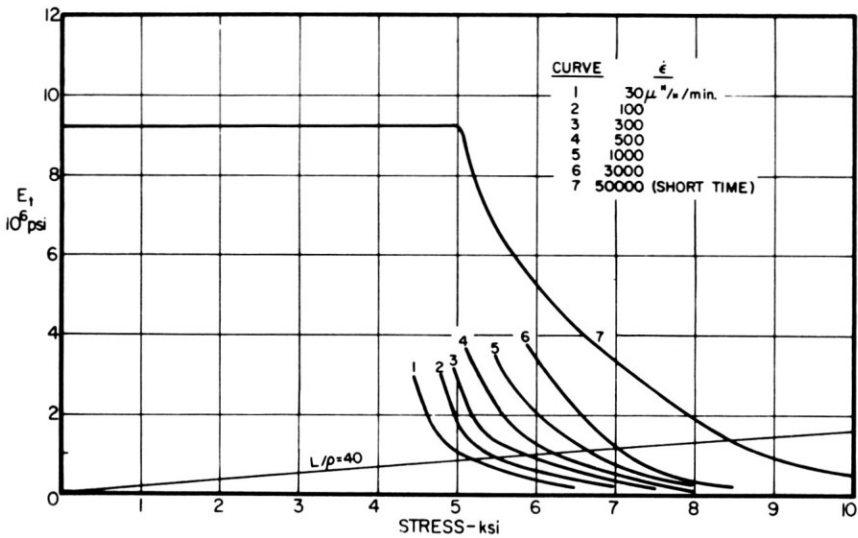


Fig. 7. Tangent modulus-stress data at constant strain rates for aluminum alloy 2024-0 at 500° F.

COLUMN EXPERIMENTS

Also shown in Fig. 8 are the results of carefully conducted creep-buckling tests on pin-ended aluminum alloy 2024-0 columns of $L/\rho = 40$ at 500°F . These results, taken from Ref. 5, are for columns that contained effective initial imperfections less than 0.005 of the thickness. The initial imperfections were deduced from Southwell analyses of the central deflection measured during the controlled application of the creep load at 500°F . The compressive creep data shown in Fig. 4 are representative of the column material.

For the test data shown in Fig. 8, the solid circles represent the times for failure of the columns whereas the open circles represent the times at which the central deflection-thickness ratio reached 0.05. This criterion, which is somewhat arbitrary, was used as a measure of the development of significant lateral deflections and is based upon the behavior of short-time column tests (shown at a time of 0.1 min. in Fig. 8). From the short-time tests, it was found that an average value of $w/t = 0.05$ corresponded to the tangent modulus column stress.

This distinction between buckling at which significant lateral deflections develop and failure is of particular importance in creep buckling, since creep is essentially a time-dependent phenomenon. In a short-time test, on the other hand, buckling and failure are essentially coincident in terms of the stress variable.

In Fig. 9, end shortening and central deflection data obtained during the course of three column creep tests at a stress level of 6,320 psi are shown. Also indicated is the theoretical critical time based on the strain rate tangent modulus at approximately one minute. It can be observed that in the neighborhood of one minute, the w/t values shown in the upper portion of Fig. 9 are close to 0.05 for two of the tests and that the corresponding end shortening data are curving upward.

These are significant indications that creep buckling has occurred in the neighborhood of the theoretical prediction for two of the three column tests.

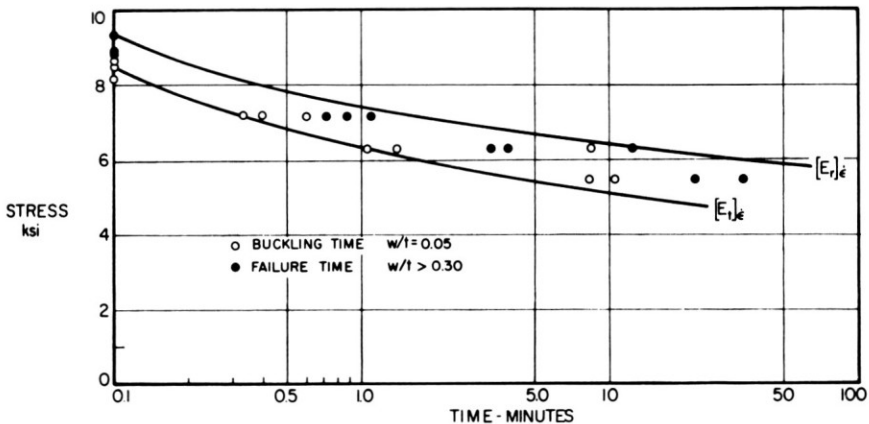


Fig. 8. Correlation of creep buckling theory and experimental data on aluminum alloy 2024-0 pin ended columns of $L/\rho = 40$ at 500°F .

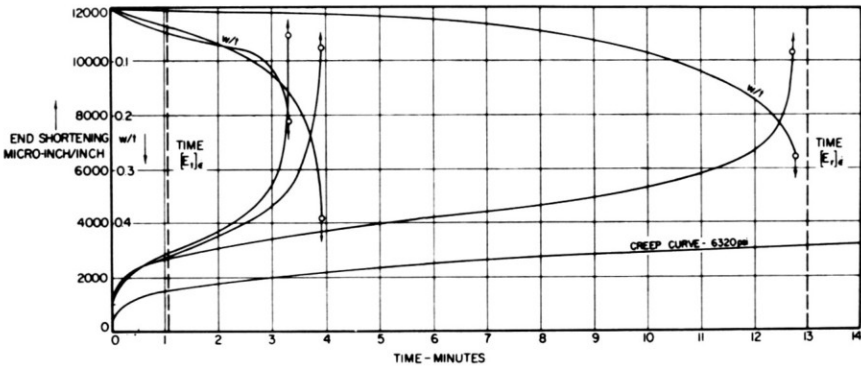


Fig. 9. End shortening and lateral deflection data for three 2024-0 aluminum alloy pin ended columns tested at a stress level of 6320 psi and 500° F.

Since the tangent modulus theory predicts the time at which significant lateral deflections can first develop for columns containing small imperfections, it does not seem particularly unusual that the third column shown in Fig. 9 reached $w/t = 0.05$ at a considerably greater time than the other two. In essence, the strain-rate dependent tangent modulus theory provides a lower bound for the development of significant lateral deflections. After buckling (at $w/t = 0.05$), considerable time can elapse before failure occurs at a w/t in the neighborhood of 0.30, as indicated in Fig. 9, as a distinctly different event.

Failure for a short-time test of an inelastic column occurs when strain reversal is complete over the convex side of the column.⁶ Consequently, in order to have some theoretical estimate of the failure time of columns for the creep case, the reduced modulus associated with strain reversal $[E_r]_i$ was calculated using the strain-rate dependent tangent modulus $[E_t]_i$. The reduced modulus values were then used in Eq. (17) in place of $[E_f]_i$. These results are shown in Figs. 8 and 9 as the curve marked $[E_r]_i$. It can be observed that the test data for buckling and failure are indeed bounded by the $[E_t]_i$ and $[E_r]_i$ curves.

CREEP BUCKLING OF PLATES AND CYLINDRICAL SHELLS

Having considered the fundamental problem of creep buckling of a column from both a theoretical and experimental viewpoint, it is pertinent now to consider creep buckling of plates and cylindrical shells. As is evident from the column problem, buckling develops at an appropriate local strain rate and at this instant the equilibrium conditions of the stability problem are satisfied.

An appropriate equilibrium equation is that for inelastic buckling of plates and shells where the secant and tangent moduli are strain-rate dependent quantities defined according to Eqs. (6) and (10). The following governing equation for inelastic buckling of an isotropic shell is based on deformation theory and was derived in Ref. 7 under the mild restriction that an external

torsional load does not act in combination with axial compression or lateral pressure.

$$\left[A_1 \frac{\partial^4}{\partial x^4} + \left(\frac{4A_1A_2}{A_3} - \frac{A_{12}^2}{A_3} - A_{12} \right) \frac{\partial^4}{\partial x^2 \partial y^2} + A_2 \frac{\partial^4}{\partial y^4} \right]$$

$$\left[A_1 \frac{\partial^4 w}{\partial x^4} + (A_{12} + A_3) \frac{\partial^4 w}{\partial x^2 \partial y^2} + A_2 \frac{\partial^4 w}{\partial y^4} + \frac{\sigma_x h}{D} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\tau h}{D} \frac{\partial^2 w}{\partial x \partial y} + \frac{\sigma_y h}{D} \frac{\partial^2 w}{\partial y^2} \right]$$

$$+ \frac{B}{R^2} \left(A_1 A_2 - \frac{A_{12}^2}{4} \right) \frac{\partial^4 w}{\partial x^4} = 0 \tag{18}$$

where:

$$A_1 = 1 - \alpha \sigma_x^2 / 4 \qquad \alpha = (3/\sigma_i^2)(1 - E_t/E_s)$$

$$A_2 = 1 - \alpha \sigma_y^2 / 4 \qquad B = 4E_s h / 3$$

$$A_{12} = 1 - \alpha \sigma_x \sigma_y / 2 \qquad D = E_s h^3 / 9$$

$$A_3 = 1 - \alpha \tau^2$$

CREEP BUCKLING OF PLATES UNDER COMPRESSION

Without any loss of generality, we can consider the case of a flat plate under an axial compressive stress, σ_a . In this case, $\sigma_y = \tau = 0, A_2 = A_{12} = A_3 = 1$ and

$$A_1 = 1/4 + (3/4)E_t/E_s \tag{19}$$

Thus, Eq. (18) reduces to the following with $R = \infty$:

$$\left(\frac{1}{4} + \frac{3 E_t}{4 E_s} \right) \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\sigma_a h}{D} \frac{\partial^2 w}{\partial x^2} = 0 \tag{20}$$

By substituting the value for E_t given by Eq. (10) into Eq. (20), and replacing E_s by $[E_s]_\epsilon$ we obtain:

$$\left(\frac{1}{4} + \frac{3 [E_t]_\epsilon}{4 [E_s]_\epsilon} \right) \frac{\partial^4 w}{\partial x^4} + \frac{3 [E_u]_\epsilon}{4 [E_s]_\epsilon} \frac{\partial^5 w}{\partial x^4 \partial t} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\sigma_a h}{D} \frac{\partial^2 w}{\partial x^2} = 0 \tag{21}$$

A solution of Eq. (21) can be given in the following form for a simply supported flat plate:

$$w = w_0 [t] \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b} \tag{22}$$

By substituting the appropriate derivatives of Eq. (22) into Eq. (21)

$$\left[\left(\frac{1}{4} + \frac{3 [E_t]_\epsilon}{4 [E_s]_\epsilon} \right) \left(\frac{m\pi}{a} \right)^4 w_0 + \frac{3 [E_u]_\epsilon}{4 [E_s]_\epsilon} \left(\frac{m\pi}{a} \right)^4 \dot{w}_0 + 2 \left(\frac{m\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 w_0 \right.$$

$$\left. + \left(\frac{\pi}{b} \right)^4 w_0 - \frac{\sigma_a h}{D} \left(\frac{m\pi}{a} \right)^2 w_0 \right] \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b} = 0 \tag{23}$$

For a nontrivial solution, the bracketed terms are set equal to zero with the following result:

$$\sigma_a = \frac{D}{h} \left[\left(\frac{1}{4} + \frac{3}{4} \frac{[E_t]_{\dot{\epsilon}}}{[E_s]_{\dot{\epsilon}}} \right) \left(\frac{m\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2 + \left(\frac{a}{m\pi} \right)^2 \left(\frac{\pi}{b} \right)^4 \right. \\ \left. + \frac{3}{4} \frac{[E_u]_{\dot{\epsilon}}}{[E_s]_{\dot{\epsilon}}} \left(\frac{m\pi}{a} \right)^2 \frac{\dot{w}_0}{w_0} \right] \quad (24)$$

For creep buckling, the Rabotnov-Shersternikov stability criterion, $w_0/w_0 = 0$ is applied to Eq. (24). As a consequence, the last term vanishes and Eq. (24) reduces to the creep-buckling analog of the inelastic buckling solution.

$$\sigma_a = \frac{\pi^2 k \eta E}{12(1-\nu_e^2)} \left(\frac{h}{b} \right)^2 \quad (25)$$

where

$$\eta = \frac{(1-\nu_e^2)}{(1-\nu^2)} \frac{[E_s]_{\dot{\epsilon}}}{E} \left[\frac{1}{2} + \frac{1}{4} \left(1 + 3 \frac{[E_t]_{\dot{\epsilon}}}{[E_s]_{\dot{\epsilon}}} \right)^{1/2} \right] \quad (26)$$

CREEP BUCKLING OF PLATES AND SHELLS

We can consider now certain aspects of the general creep-buckling problem that become apparent from the analysis of the flat plate under axial compression. The essential result of applying the Rabotnov-Shersternikov stability criterion, $w_0/w_0 = 0$, is that the time-dependent term in the definition of the tangent modulus vanishes so that Eq. (10) reduces to

$$E_t = [E_t]_{\dot{\epsilon}} = \delta\sigma_i / \delta\epsilon_i \quad (27)$$

As a consequence, the creep-buckling problem becomes quasi-static as pointed out in Ref. 1.

As defined by Eq. (6), the secant modulus is

$$[E_s]_{\dot{\epsilon}} = \sigma_i / \epsilon_i \quad (28)$$

Therefore, the strain-rate dependent tangent and secant moduli for a fixed stress in the creep-buckling problem are directly analogous to the tangent and secant moduli associated with the increasing stress in the inelastic buckling problem.

As a consequence, the plasticity reduction factors obtained for inelastic buckling of flat plates and shells under various types of loadings may be used directly in the creep-buckling problem with the understanding that the tangent and secant moduli are strain-rate dependent for the latter. Thus, the inelastic buckling results obtained in Ref. 8 for flat plates, in Ref. 9 for shells and in Ref. 6 for orthotropic plates and cylindrical shells may be employed for creep buckling of materials with arbitrary creep characteristics by following a graphical procedure similar to that presented for columns.

EXPERIMENTS ON PLATES AND SHELLS

The creep-buckling theory presented herein predicts the time at which significant lateral deflections first develop. Under short-time loading it is well known that the failure of plates and many classes of shells occurs as a distinctly later event than buckling. Thus, the separation in time between creep buckling and creep collapse can be expected to be even more pronounced for plates and many classes of shells than for the column case discussed previously.

As a consequence, it is important to evaluate the predictions of creep-buckling theory by conducting carefully controlled tests on plates and shells in a manner similar to that used for the column tests (Fig. 9). In such tests, it is necessary to measure both lateral deflection and end shortening as a function of time and to obtain independently the pertinent compressive creep properties of the material used. The lateral deflection measurements indicate when significant lateral deflections develop and when failure occurs. By comparing the end shortening and compressive creep data, the presence of unsuspected frictional effects during the creep test which may be contributed by the plate supporting jig, for example, can be detected.

At the time of preparation of this paper, the few published data available on plates and shells did not satisfy these criteria. Consequently, it is not possible to evaluate the creep-buckling theory in a critical manner. Such tests are in progress as part of our current program and it is hoped that they can confront the theory with satisfactory test data. (See Ref. 10)

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